FOUR - TERMINAL NETWORKS

Electrical circuit having four terminals is called four-terminal network or two-port. In some cases, when one takes four-terminal network into consideration as separated element, as example acoustic, electromechanical, optic devices etc., such network can be called as quadripole. Two terminals or poles are input, and another two - output. Each four-terminal network can be investigated using common circuit theory laws such as Ohm’s, Kirchoff’s and calculating mesh currents or nodal voltages. Four-terminal networks theory is the case of adaptation of these laws to take into consideration only input and output variables.

Primary parameters of the four-terminal networks. Symbol of the four-terminal network and possible directions of the voltages and currents are shown in the Fig. 3.1. Properties of the four-terminal network depend on relation between input and output voltages and currents. Usually two of voltages or currents are known and are called excitation or action. Another two voltages or currents are called response or reaction into these excitations. Relation between these voltages and currents depends on four-terminal properties. To build equations for these relations means to express reaction voltage or current in term of action voltage or current. There are six combinations of designation off excitations and reactions in the four-terminal networks. So voltages and currents may be expressed using six equation systems. Coefficients in these equations are called primary parameters of the network. All these parameters are shown in the table 3.1.

Table 3.1

<table>
<thead>
<tr>
<th>Version</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Action variables</td>
<td>( \vec{I}_1, \vec{I}_2 )</td>
<td>( \vec{V}_1, \vec{V}_2 )</td>
<td>( i_1, i_2 )</td>
<td>( \vec{V}_1, \vec{I}_2 )</td>
<td>( \vec{V}_1, \vec{I}_2 )</td>
<td>( \vec{V}_1, \vec{I}_2 )</td>
</tr>
<tr>
<td>Reaction variables</td>
<td>( \vec{V}_1, \vec{V}_2 )</td>
<td>( \vec{I}_1, \vec{I}_2 )</td>
<td>( i_1, i_2 )</td>
<td>( \vec{I}_1, \vec{I}_2 )</td>
<td>( i_1, i_2 )</td>
<td>( i_1, i_2 )</td>
</tr>
<tr>
<td>System of parameters</td>
<td>Z ( \begin{bmatrix} z_{11} &amp; z_{12} \ z_{21} &amp; z_{22} \end{bmatrix} )</td>
<td>Y ( \begin{bmatrix} y_{11} &amp; y_{12} \ y_{21} &amp; y_{22} \end{bmatrix} )</td>
<td>H ( \begin{bmatrix} h_{11} &amp; h_{12} \ h_{21} &amp; h_{22} \end{bmatrix} )</td>
<td>A ( \begin{bmatrix} a_{11} &amp; a_{12} \ a_{21} &amp; a_{22} \end{bmatrix} )</td>
<td>G ( \begin{bmatrix} g_{11} &amp; g_{12} \ g_{21} &amp; g_{22} \end{bmatrix} )</td>
<td>B ( \begin{bmatrix} b_{11} &amp; b_{12} \ b_{21} &amp; b_{22} \end{bmatrix} )</td>
</tr>
</tbody>
</table>

From these equations becomes clear mean of primary parameters. First two sets of equations relate network voltages and currents. Their coefficients as in Ohm’s law have dimension of impedance or admittance. So why coefficients in the first equation are marked as \( z \)-parameters and can be called impedance parameters. In the second equation coefficients are denoted as \( y \)-parameters and can be called as admittance parameters and they are inverse of impedance parameters in most cases. In the third equation we have mixture of variables. Such parameters were called \( h \)-parameters because of they are hybrid. As you had learned in the course of electronics fundamentals such mixture of variables actually arises as a simplification of mid-frequency model of a common emitter configuration of bipolar transistor. Coefficients of the fourth equation relate output variables with input variables and are called transmission parameters or \( t \)-parameters. In some textbooks these parameters can be called as \( a \)-parameters or ABCD parameters. Last two types of equations also are mixtures of variables. As you can see
they are similar to third and fourth equations and have exchanged action and reaction variables: 
\[ \dot{V}_1 \leftrightarrow \dot{V}_2, \quad \dot{I}_1 \leftrightarrow \dot{I}_2. \]

We know that voltage and current phasors are complex quantities. So coefficients in the equations above could be complex values also. As we’ll see later, in some cases these coefficients could be real values.

Usually only four types of primary four-terminal network parameters – impedance, admittance, hybrid and transmission - are used in circuit theory.

Scalar notation of equations in terms of primary parameters, voltages and currents in the four-terminal networks it is possible to change into matrix form:

\[
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2
\end{bmatrix}
= 
\begin{bmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{I}_1 \\
\dot{I}_2
\end{bmatrix},
\begin{bmatrix}
\dot{I}_1 \\
\dot{I}_2
\end{bmatrix}
= 
\begin{bmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2
\end{bmatrix}.
\]

(3.2)

Matrix form is more convenient in the analysis of the complicated network, consisting on some simple four-terminals.

Let’s consider two four-terminals in series (Fig. 3.2) as example. If we’ll choose directions of the currents and voltages as shown in the Fig. 3.2, then

\[
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2
\end{bmatrix}
= 
\begin{bmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{I}_1 \\
\dot{I}_2
\end{bmatrix},
\begin{bmatrix}
\dot{I}_1 \\
\dot{I}_2
\end{bmatrix}
= 
\begin{bmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2
\end{bmatrix}.
\]

Because of left-side network’s output voltage and current phasors are the same as for the right-side network’s input voltage and current phasors

\[
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2
\end{bmatrix}
= 
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2
\end{bmatrix}.
\]

Here \( |T_{11}| = |T_1| |T_2| \) - transmission coefficient’s matrix of the equivalent four-terminal network, which represents two in series, connected networks.

Now let’s consider two four-terminal networks connected in parallel (Fig. 3.3a). If they are described using \( y \)-parameters, input and output currents and voltages in such network have following relationships:

\[
\begin{bmatrix}
\dot{I}_1 \\
\dot{I}_2
\end{bmatrix}
= 
\begin{bmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2
\end{bmatrix},
\begin{bmatrix}
\dot{V}_1 \\
\dot{V}_2
\end{bmatrix}
= 
\begin{bmatrix}
y_{11} & |Y_1| + |Y_2|
\end{bmatrix}
\begin{bmatrix}
\dot{I}_1 \\
\dot{I}_2
\end{bmatrix}.
\]

When two parallel four-terminal networks are connected as shown in the Fig. 3.3b, use of the \( z \)-parameters yields to relationships...
General relations between primary parameters of the four-terminal networks. All types of four-terminal network’s primary parameters are interrelated. It is rather simple to express one type of parameters in the terms of another type of parameters. Let’s try to express $z$-parameters through $t$-parameters as example. At the beginning it is necessary to note that the goal of such expression is to obtain $\hat{V}_1$ and $\hat{V}_2$ in terms of $I_1$ and $I_2$. So it is necessary to do some transforms of initial equations whose relates external variables in four-terminal network and network $t$-parameters. At first let’s recall initial system of equations with $t$-parameters:

\[
\begin{align*}
\bar{V}_1 &= t_{11} \hat{V}_2 + t_{12} \hat{I}_2, \\
\bar{I}_1 &= t_{21} \hat{V}_2 + t_{22} \hat{I}_2.
\end{align*}
\]

Transforming second equation and substituting value of into the first equation we are obtaining:

\[
\begin{align*}
\hat{V}_2 &= \frac{I_1 - t_{22} \hat{I}_2}{t_{21}}, \\
\hat{V}_1 &= t_{11} \hat{I}_1 - t_{11} t_{22} \hat{I}_2 + t_{12} \hat{I}_2.
\end{align*}
\]

From these equations follows:

\[
\begin{align*}
\hat{V}_1 &= \frac{t_{11} \hat{I}_1 + \left(t_{12} - \frac{t_{11} t_{22}}{t_{21}}\right) \hat{I}_2}{t_{21}} = z_{11} \hat{I}_1 + z_{12} \hat{I}_2 \\
\hat{V}_2 &= \frac{1}{t_{21}} \hat{I}_1 - \frac{t_{22}}{A_{21}} \hat{I}_2 = z_{21} \hat{I}_1 + z_{22} \hat{I}_2
\end{align*}
\]

Therefore,

\[
\begin{align*}
z_{11} &= \frac{t_{11}}{t_{21}}; \quad z_{12} = t_{12} - \frac{t_{11} t_{22}}{t_{21}}; \quad z_{21} = \frac{1}{t_{21}}; \quad z_{22} = -\frac{t_{22}}{t_{21}}.
\end{align*}
\]

So, in the same manner it is possible to relate all systems of primary parameters.

Basic simple four-terminal networks and their primary parameters. Four-terminal network primary parameters depend on real circuit parameters. In network theory are well known some basic simple circuits, shown in the figure 3.4.

Parameters at these circuits could be calculated in easy way. As example we’ll show calculation of the $[T]$ parameters for first three circuits. For this we shall denote directions of the currents and voltages as shown in the Fig. 3.4.

At first let’s try to consider first circuit (Fig.3.4a). In this circuit

\[
\begin{align*}
\hat{I}_1 = \hat{I}_2 = t_{21} \hat{V}_2 + t_{22} \hat{I}_2, \\
\hat{V}_1 = \hat{I}_2 Z + \hat{V}_2 = t_{11} \hat{V}_2 + t_{12} \hat{I}_2.
\end{align*}
\]
Therefore \( t_{11} = 1; \ t_{12} = Z; \ t_{21} = 0; \ t_{22} = 1 \).

For the second circuit

\[
\begin{align*}
\dot{V}_1 &= \dot{V}_2, \\
\dot{I}_1 &= \dot{V}_2/Z + I_2.
\end{align*}
\]

So, from these equations follows that \( t_{11} = 1; \ t_{12} = 0; \ t_{21} = 1/Z; \ t_{22} = 1 \).

For the third circuit

\[
\begin{align*}
\dot{V}_1 &= I_1Z_1 + \dot{V}_2, \\
\dot{I}_1 &= I_2 + \dot{V}_2/Z_2.
\end{align*}
\]

\[
\begin{align*}
\dot{V}_1 &= \left( \dot{I}_2 + \frac{\dot{V}_2}{Z_2} \right)Z_1 + \dot{V}_2 = \dot{V}_2\left( 1 + \frac{Z_1}{Z_2} \right) + I_2Z_1, \\
\dot{I}_1 &= \dot{I}_2 + \dot{V}_2/Z_2.
\end{align*}
\]

So \( t_{11} = 1 + Z_1/Z_2; \ t_{12} = Z_1; \ t_{21} = 1/Z_2; \ t_{22} = 1 \).

Parameters of the third circuit could be obtained multiplying transmission matrixes \( T \) of the first and second circuits also:

\[
|T| = \begin{bmatrix} 1 & Z_1 \\ 0 & 1/Z_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/Z_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{Z_1}{Z_2} & Z_1 \\ \frac{1}{Z_2} & 1 \end{bmatrix}.
\]

Let’s try to derive t-coefficients for the fifth circuit. At first we can write two equations, which relate voltages and current in the circuit in terms of circuit impedance:
\[
\begin{align*}
\dot{V}_1 &= \dot{V}_2 + i_{12}Z_2 = \dot{V}_2 + \left(\frac{\dot{V}_2}{Z_3} + i_2\right)Z_2, \\
I_1 &= \frac{V_1}{Z_1} + I_{12} = \frac{V_1}{Z_1} + I_2 + \frac{\dot{V}_2}{Z_3}.
\end{align*}
\]

Simplifying input voltage phasors expression
\[
\dot{V}_1 = \dot{V}_2 \left(1 + \frac{Z_2}{Z_3}\right) + i_2Z_2
\]
and substituting it into input current phasor expression
\[
\dot{I}_1 = \dot{V}_2 \left(1 + \frac{Z_2}{Z_3}\right) + i_2Z_2 + \frac{\dot{V}_2}{Z_3} + i_2 = \dot{V}_2 \left(\frac{1}{Z_3} + \frac{1}{Z_1} + \frac{Z_2}{Z_1Z_3}\right) + i_2 \left(1 + \frac{Z_2}{Z_1}\right)
\]
we are able to obtain final expressions of t-parameters:
\[
t_{11} = 1 + \frac{Z_2}{Z_3}; \quad t_{12} = Z_2; \quad t_{21} = \frac{1}{Z_3} + \frac{1}{Z_1} + \frac{Z_2}{Z_1Z_3}; \quad t_{22} = 1 + \frac{Z_2}{Z_1}.
\]

In the same manner we can write equations for the sixth network
\[
\begin{align*}
\dot{V}_1 &= I_1Z_1 + i_2Z_3 + \dot{V}_2; \\
\dot{I}_1 &= \dot{I}_2 + (\dot{V}_2 + i_2Z_2)/Z_3.
\end{align*}
\]

Transforming these equations yields
\[
\begin{align*}
\dot{I}_1 &= \dot{V}_2 / Z_3 + \dot{I}_2 (1 + Z_2 / Z_3); \\
\dot{V}_1 &= (\dot{V}_2 / Z_3 + \dot{I}_2 (1 + Z_2 / Z_3))Z_1 + \dot{I}_2Z_3 + \dot{V}_2 = \dot{V}_2 (1 + Z_1 / Z_3) + \dot{I}_2 (Z_1 + Z_3 + Z_1Z_2 / Z_3).
\end{align*}
\]

So, t-coefficients could be found as follows:
\[
t_{11} = 1 / Z_3; \quad t_{12} = Z_1 + Z_3 + Z_1Z_2 / Z_3; \quad t_{21} = 1 / Z_3; \quad 1 + Z_2 / Z_3.
\]

**Experimental definition of the primary parameters.** Primary parameters of the four-terminal networks could be found experimentally, then one of the excitations (input variables) is equal to zero. Such conditions can be obtained in open circuit (then current is equal to zero) or shorted circuit (then voltage is equal to zero). Let’s try to find in such way primary parameters of the four-terminal network.

If network output is open, then \( \dot{I}_2 = 0 \) and network equations could be written in more simple form:
\[
\begin{align*}
\dot{V}_1 &= t_{11}\dot{V}_2, & \dot{V}_1 &= z_{11}\dot{I}_1, & \dot{I}_1 &= t_{21}\dot{V}_2, & \dot{V}_2 &= z_{21}\dot{I}_1.
\end{align*}
\]

From these equations follows:
\[ t_{11} = \frac{\dot{V}_1}{\dot{V}_2} \bigg|_{I_2=0} ; \quad z_{11} = \frac{\dot{V}_1}{\dot{I}_2} \bigg|_{I_2=0} ; \quad t_{21} = \frac{\dot{I}_2}{\dot{V}_2} \bigg|_{I_2=0} ; \quad z_{21} = \frac{\dot{V}_2}{\dot{I}_1} \bigg|_{I_2=0}. \]

When network output is shorted (\( \dot{V}_2 = 0 \)), then

\[ \dot{V}_1 = t_{12} \dot{I}_2, \quad \dot{I}_1 = y_{11} \dot{V}_1, \quad \dot{V}_1 = h_{11} \dot{I}_1, \quad \dot{I}_1 = t_{21} \dot{I}_2, \quad \dot{I}_2 = y_{21} \dot{V}_1, \quad \dot{I}_2 = h_{21} \dot{I}_1. \]

So

\[ t_{12} = \frac{\dot{V}_1}{\dot{I}_2} \bigg|_{I_2=0} ; \quad y_{11} = \frac{\dot{I}_1}{\dot{V}_1} \bigg|_{I_2=0} ; \quad h_{11} = \frac{\dot{V}_1}{\dot{I}_1} \bigg|_{I_2=0} ; \quad t_{21} = \frac{\dot{I}_1}{\dot{I}_2} \bigg|_{I_2=0} ; \quad y_{21} = \frac{\dot{I}_2}{\dot{V}_1} \bigg|_{I_2=0} ; \quad h_{21} = \frac{\dot{I}_2}{\dot{I}_1} \bigg|_{I_2=0}. \]

When network input is open (\( \dot{I}_1 = 0 \)), then

\[ \dot{V}_1 = z_{12} \dot{I}_2; \quad \dot{V}_1 = h_{12} \dot{V}_2; \quad \dot{V}_2 = z_{22} \dot{I}_2; \quad \dot{I}_2 = h_{22} \dot{V}_2. \]

Therefore

\[ z_{12} = \frac{\dot{V}_1}{\dot{I}_2} \bigg|_{I_2=0} , \quad h_{12} = \frac{\dot{V}_1}{\dot{V}_2} \bigg|_{I_2=0} , \quad z_{22} = \frac{\dot{V}_2}{\dot{I}_2} \bigg|_{I_2=0} , \quad h_{22} = \frac{\dot{I}_2}{\dot{V}_2} \bigg|_{I_2=0}. \]

And when network input is shorted (\( \dot{V}_1 = 0 \)), then

\[ \dot{I}_1 = y_{12} \dot{V}_2; \quad \dot{I}_2 = y_{22} \dot{V}_2. \]

So

\[ y_{12} = \frac{\dot{I}_1}{\dot{V}_2} \bigg|_{I_2=0} ; \quad y_{22} = \frac{\dot{I}_2}{\dot{V}_2} \bigg|_{I_2=0}. \]

All these formulas of the primary parameters could be used as mathematical definitions. These formulas also show physical meaning of the primary parameters.

For example, four-terminal network primary parameter

\[ t_{11} = \frac{\dot{V}_1}{\dot{V}_2} \bigg|_{I_2=0} = \frac{1}{\dot{V}_2 \bigg|_{I_2=0}} = \frac{1}{H_V \bigg|_{I_2=0}}. \]

is reverse function of the voltage transfer function in the open four-terminal network. Parameter \( t_{21} \) relates input current and output voltage in the open network. It can be called feedback-coupling admittance. \( t_{12} \) – shorted network crossing (direct coupling) impedance, \( t_{22} \) – reverse function of the current transfer in the shorted network. Also we can see that all \( z \)-parameters has impedance dimension and \( y \)-parameters – admittance dimension. For example \( z_{11} \) – input impedance of the open network, \( z_{21} \) – direct coupling impedance, \( z_{12} \) – network with open input.
feedback coupling resistance, $z_{22}$—output impedance of the network with the open input. For the networks described using $h$-parameters - $h_{11}$—input impedance of the open network, $h_{21}$—current transfer coefficient for the open circuit, $h_{22}$—output admittance of the circuit with the open input and $h_{12}$—voltage feedback coefficient in the network with open input.

**Equivalent circuits of the four-terminal networks.** Usually four-terminal network represents complicated circuit consisting on a lot of various components. Using various transforms such circuit can be simplified and expressed as canonic circuit with minimal amount of passive components. Minimal amount of independent components in linear circuit is equal to three. So, each four-terminal network could be represented as $T$ or $\Pi$ circuit (Fig. 3.5), consisting on three impedances. These impedances can be expressed in terms of four-terminal network primary parameters. Let’s try to express impedances of the $T$ circuit through primary $t$-parameters as example. At first let’s recall equations which relates $t$-parameters and circuit impedances:

$$
Z_1 = \frac{t_{11} - 1}{t_{21}}; \quad Z_2 = \frac{1}{t_{21}}; \quad Z_3 = \frac{t_{22} - 1}{t_{21}}.
$$

In the same manner we can write expressions of $t$-parameters for the $\Pi$-type circuit:

$$
t_{11} = 1 + \frac{Z_2}{Z_3}; \quad t_{12} = Z_2; \quad t_{21} = \frac{1}{Z_1} + \frac{Z_2}{Z_1 Z_3}; \quad t_{22} = 1 + \frac{Z_2}{Z_1}.
$$

Therefore

$$
Z_1 = \frac{t_{12}}{t_{22} - 1}; \quad Z_2 = t_{12}; \quad Z_3 = \frac{t_{12}}{t_{11} - 1}.
$$

So, having primary parameters we always can to construct equivalent $T$ or $\Pi$ circuit.

Now let’s consider another type of the four-terminal network equivalent circuits. As you learned in electronics fundamentals course, model of a common emitter configuration of bipolar transistor consist on two depended sources, impedance and admittance (Fig.3.7a). These elements could be described using $h$-parameters. Such type of model rests with the interpretation of appropriate mathematical equations:

$$
\begin{align*}
\dot{V}_1 &= h_{11}\dot{I}_1 + h_{12}\dot{V}_2, \\
\dot{I}_2 &= h_{21}\dot{I}_1 + h_{22}\dot{V}_2.
\end{align*}
$$
Model’s input voltage $V_1$ equals the sum of two voltages; $h_{11}I_1$ plus the voltage due to a voltage-controlled voltage source given by $h_{12}V_2$. This is precisely the left-hand portion of figure 3.7a. A similar interpretation follows for the right-hand side of figure 3.7a, here current $I_2$ equals to current due to current controlled current source $h_{21}I_1$ plus current $h_{22}V_2$. Such type of four-terminal representation is called as two-depended source equivalent circuit. The similar two-depended source equivalent circuits can be build for the four-terminal networks equated by $z$, $y$ or $t$ parameters (Fig.3.7).

It is possible to build one-depended source equivalent circuits of the four-terminal circuits also. Transforming one depended source of the earlier discussed circuits in to the direct coupling impedance we can obtain two types of one-depended source equivalent circuits shown on the Fig.3.8.

Let’s try to relate parameters of these circuits with $z$-parameters and $y$-parameters of the four-terminal networks. Let’s start from the circuit shown in the Fig. 3.8a. The mesh equations for this circuit yields

\[
\begin{aligned}
    \dot{V}_1 &= Z_1I_1 + Z_2(I_1 + I_2), \\
    \dot{V}_2 &= Z_3I_2 + Z_2(I_1 + I_2) + \dot{E}.
\end{aligned}
\]

The next phase is to transform initial $z$-parameters equations:

\[
\begin{aligned}
    \dot{V}_1 &= z_{11}I_1 + z_{12}I_2 = z_{11}I_1 + z_{12}I_1 + z_{12}I_1 - z_{12}I_1 = (z_{11} - z_{12})I_1 + z_{12}(I_1 + I_2), \\
    \dot{V}_2 &= z_{21}I_1 + z_{22}I_2 = z_{21}I_1 + z_{22}I_2 + z_{12}(I_1 + I_2) - z_{13}(I_1 + I_2) = \\
    &= (z_{22} - z_{12})I_2 + z_{12}(I_1 + I_2) + (z_{21} - z_{12})I_1.
\end{aligned}
\]

Here, comparing these equations with the previous formula, we see that

\[
\begin{aligned}
    Z_1 &= z_{11} - z_{12}; \\
    Z_2 &= z_{12}; \\
    Z_3 &= z_{22} - z_{12}; \\
    \dot{E} &= (z_{21} - z_{12})I_1.
\end{aligned}
\]

In the same manner it is possible to derive that circuit’s on Fig.3.8b admittance and voltage controlled current source may be expressed through $y$-parameters. Nodal equations

\[
\begin{aligned}
    \dot{I}_1 &= Y_1\dot{V}_1 + Y_2(\dot{V}_1 - \dot{V}_2); \\
    \dot{I}_2 &= Y_3\dot{V}_1 + Y_2(\dot{V}_1 - \dot{V}_2) + \dot{J},
\end{aligned}
\]

\[
\begin{aligned}
    \dot{I}_1 &= y_{11}\dot{V}_1 + y_{12}\dot{V}_2 = y_{11}\dot{V}_1 + y_{12}\dot{V}_1 + y_{12}\dot{V}_1 - y_{12}\dot{V}_1 = (y_{11} + y_{12})\dot{V}_1 - y_{12}(\dot{V}_1 - \dot{V}_2); \\
\end{aligned}
\]

As in previous equivalent circuit let’s transform initial $y$-parameters equations:
\[
\begin{align*}
\dot{I}_1 &= y_{11}\dot{V}_1 + y_{12}\dot{V}_2 = y_{11}\dot{V}_1 + y_{12}\dot{V}_2 + y_{12}\dot{V}_1 - y_{12}\dot{V}_2 = (y_{11} + y_{12})\dot{V}_1 - y_{12}(\dot{V}_1 - \dot{V}_2), \\
\dot{I}_2 &= y_{21}\dot{V}_1 + y_{22}\dot{V}_2 = y_{21}\dot{V}_1 + y_{22}\dot{V}_2 + y_{12}(\dot{V}_1 - \dot{V}_2) - y_{12}(\dot{V}_1 - \dot{V}_2) = \\
&= (y_{22} + y_{12})\dot{V}_2 + y_{12}(\dot{V}_1 - \dot{V}_2) + \dot{V}_2 (y_{21} - y_{22})
\end{align*}
\]

\[ Y_1 = y_{11} + y_{12}; \quad Y_2 = -y_{12}; \quad Y_3 = y_{22} + y_{12}; \quad \dot{J} = (y_{21} - y_{12})\dot{V}_2. \]

**Secondary parameters of the four-terminal networks.** Such parameters are useful and important for determining power transfer and various gain computations. In four-terminal networks theory are known following secondary parameters:

**Input impedance**

\[ Z_{in} = \frac{\dot{V}_i}{I_1} = \frac{1}{Y_{in}}. \]

**Output impedance**

\[ Z_{out} = \frac{\dot{V}_{2o}}{I_{2s}}. \]

Here \( \dot{V}_{2o} \) – output voltage phasor in the open circuit and \( I_{2s} \) - output current phasor in the shorted circuit.

**Voltage gain or complex voltage transfer function**

\[ \dot{H}_v = \frac{\dot{V}_2}{\dot{V}_1}. \]

**Current gain or complex current transfer function**

\[ \dot{H}_i = \frac{\dot{I}_2}{\dot{I}_1}. \]

**Transfer (direct coupling) impedance**

\[ Z = \frac{\dot{V}_2}{\dot{I}_1} = \dot{H}_v Z_L. \]

**Transfer (direct coupling) admittance**

\[ Y = \frac{\dot{I}_2}{\dot{V}_1} = \frac{\dot{H}_v}{Z_L}. \]

**Characteristic and wave impedances.** Definition of these impedances will be given below.

**Propagation constant** is parameter used in circuit theory as alternative for four-terminal network voltage or current gain. This constant can be denoted as \( g \). Four-terminal network output voltage or current using this parameter could be calculated as follows:

\[ \dot{V}_2 = \dot{V}_1 e^{-g}, \]
\[ \dot{I}_2 = \dot{I}_1 e^{-g}. \]

It is necessary to mention that equality between voltage and current expressions can be reached only in particular cases.

Let’s consider all these parameters in more detailed form. Expression of the four-terminal network secondary parameters depends on type of its primary parameters and network equivalent circuit. Let’s try to show methods of determination of secondary parameters using \( t \)-parameters and two-depended source equivalent circuits.

**Input impedance** can be easy calculated in terms of \( t \)-parameters:
Using the terminal condition by the load impedance $Z_L$ we obtain $\dot{I}_2 = \dot{V}_2 / Z_L$. Incorporating this terminal condition into previous equation yields

$$Z_{in} = \frac{\dot{V}_1}{\dot{I}_1} = \frac{t_{11}\dot{V}_2 + t_{12}\dot{I}_2}{t_{21}\dot{V}_2 + t_{22}\dot{I}_2}.$$ 

If network output is shorted ($Z_L = 0$), then input impedance $Z_{ins} = \frac{t_{12}}{t_{22}}$.

If network output is open ($Z_L = \infty$), then input impedance $Z_{ino} = \frac{t_{11}}{t_{21}}$.

**Output impedance.** At first we must find $\dot{V}_{2o}$ and $\dot{I}_{2o}$. In the open network ($Z_L = \infty, \ \dot{I}_2 = 0$)

$$\dot{V}_1 = t_{11}\dot{V}_{2o}; \quad \dot{I}_1 = t_{21}\dot{V}_{2o}.$$ 

Input phasor voltage could be found from following equation:

$$\dot{V}_1 = \dot{V}_S - \dot{I}_1 Z_S.$$ 

Here $\dot{V}_S$ – source voltage phasor, $Z_S$ – source internal resistance. Comparison of these equations produces

$$t_{11}\dot{V}_{2o} = \dot{V}_S - Z_S t_{21}\dot{V}_{2o} \quad \text{and} \quad \dot{V}_{2o} = \frac{\dot{V}_S}{t_{11} + Z_S t_{21}}.$$ 

In the shorted network $Z_L = 0, \dot{V}_2 = 0$ and

$$\dot{V}_1 = t_{12}\dot{I}_{2s} = \dot{V}_S - \dot{I}_1 Z_S; \quad \dot{I}_1 = t_{22}\dot{I}_{2s}.$$ 

From these equations follows that

$$t_{12}\dot{I}_{2s} = \dot{V}_S - Z_S t_{22}\dot{I}_{2s} \quad \text{and} \quad \dot{I}_{2s} = \frac{\dot{V}_S}{t_{12} + Z_S t_{22}}.$$ 

Incorporating of $\dot{V}_{2o}$ and $\dot{I}_{2s}$ in to primary expression of output impedance yields

$$Z_{out} = \frac{t_{22}Z_S + t_{12}}{t_{21}Z_S + t_{11}}.$$ 

If voltage source is ideal ($Z_S = 0$), such network output impedance $Z_{out s} = \frac{t_{12}}{t_{11}}$.

In the same manner it is possible to prove that four-terminal network output impedance expressed in terms of $t$-parameters
$Z_{\text{out}} = \frac{t_{12}}{t_{11}}.$

if ideal current source ($Z_i = \infty$) is acting in the input of such network.

**Voltage gain or transfer coefficient.**

Equation, which relates input voltage phasor and output voltage phasor in terms of t-coefficients

$$\dot{V}_1 = t_{11} \dot{V}_2 + t_{12} \dot{I}_2$$

we can transform incorporating of terminal condition

$$\dot{I}_2 = \dot{V}_2 / Z_L :$$

$$\dot{V}_1 = (t_{11} + t_{12} / Z_L) \dot{V}_2 .$$

So, voltage transfer function

$$K_V = \frac{\dot{V}_2}{\dot{V}_1} = \frac{1}{t_{11} + t_{12} / Z_L} .$$

**Current gain or current transfer coefficient** can be derived in terms of network t-parameters in the same manner as voltage gain:

$$K_I = \frac{1}{t_{21} Z_L + t_{22}} .$$

Now let’s consider voltage gain as ratio $\dot{V}_2 / \dot{V}_S$ (Fig. 3.9). From t-parameters equation with terminal conditions

$$\begin{cases} \dot{V}_1 = t_{11} \dot{V}_2 + t_{12} \dot{I}_2 = \dot{V}_2 \left( t_{11} + t_{12} / Z_L \right) , \\ \dot{I}_1 = t_{21} \dot{V}_2 + t_{22} \dot{I}_2 = \dot{V}_2 \left( t_{21} + t_{22} / Z_L \right) , \\ \dot{V}_1 = \dot{V}_S - \dot{I}_1 Z_S . \end{cases}$$

follows:

$$\dot{V}_2 \left( t_{11} + t_{12} / Z_L \right) = \dot{V}_S - Z_S \left( t_{21} + t_{22} / Z_L \right) ; \quad \dot{V}_2 \left( t_{11} + t_{12} / Z_L + Z_L t_{21} + t_{22} Z_S / Z_L \right) = \dot{V}_S \quad \text{and}$$

$$K_V = \frac{\dot{V}_2}{\dot{V}_S} = \frac{Z_L}{t_{12} + t_{11} Z_L + t_{22} Z_S + t_{21} Z_S Z_L / Z_L} = \frac{Z_L}{t_{12} + t_{11} Z_L + Z_S (t_{22} + t_{21} Z_L)} .$$

**Characteristic and wave impedance.** It is known in the four-terminal networks theory that there is two impedance $Z_{01}$ and $Z_{02}$ with following properties:

1. Network input impedance is equal to $Z_{01}$ then network is loaded with impedance $Z_{02}$.
2. Network output impedance is equal to $Z_{02}$ then source internal resistance is equal to $Z_{01}$. 
These impedances are called characteristic impedances. With respect to definition of input and output impedances we can write:

\[ Z_{01} = \frac{t_{11}Z_{02} + t_{12}}{t_{21}Z_{02} + t_{22}}; \quad Z_{02} = \frac{t_{22}Z_{01} + t_{12}}{t_{21}Z_{01} + t_{11}}. \]

Solving these equations yields

\[ Z_{01} = \sqrt{\frac{t_{11}t_{12}}{t_{21}t_{22}}}; \quad Z_{02} = \sqrt{\frac{t_{12}t_{22}}{t_{11}t_{11}}}. \]

These equations can be rewritten in following manner:

\[ Z_{01} = \sqrt{Z_{in}Z_{in}}; \quad Z_{02} = \sqrt{Z_{out}Z_{out}}. \]

Therefore, these equations can be used to define characteristic impedances in experimental way. In the symmetric networks parameters \( t_{11} = t_{22} \) and both characteristic impedances are equal also:

\[ Z_{01} = Z_{02} = \frac{t_{12}}{t_{21}} = Z_0. \]

Characteristic impedance \( Z_0 \) is called wave impedance in such type of the four-terminal networks.

Network loaded by impedance equal to its wave impedance is called matched network. Input impedance in the matched network is equal to its wave impedance. So matched networks are acting as impedance transformers.

Let’s consider voltage and current transfer functions in the four-terminal network. Suppose that network is matched. Inspecting terminal conditions and characteristic impedance definition yields

\[ Z_L = \frac{V_2}{I_2} = Z_{02} = \frac{t_{12}t_{22}}{t_{11}t_{21}}. \]

Recalling network equations with t-parameters

\[ \dot{V}_1 = t_{11}\dot{V}_2 + t_{12}\dot{I}_2, \]
\[ \dot{I}_1 = t_{22}\dot{V}_2 + t_{21}\dot{I}_2, \]

and substituting previous equation we obtain:

\[ \dot{H}_y = \frac{\dot{V}_1}{V_2} = \frac{\dot{V}_2}{V_2} - t_{12} = t_{11} + \sqrt{\frac{t_{11}t_{22}t_{12}}{t_{22}}}; \quad \dot{H}_x = \frac{\dot{I}_1}{I_2} = \frac{\dot{I}_2}{I_2} - t_{21} = t_{22} + \sqrt{\frac{t_{12}t_{21}t_{22}}{t_{11}}}. \]
In the symmetric networks \( t_{11} = t_{22} \) and

\[
\frac{\dot{V}_1}{V_2} = \frac{\dot{I}_1}{I_2} = t_{11} + \sqrt{t_{12}t_{21}}.
\]

This ratio could be marked as value \( e^g \), where \( g \) – network propagation constant. From this definition follows that

\[
e^g = \frac{\dot{V}_1}{V_2} = \frac{\dot{I}_1}{I_2}
\]

and

\[
g = \ln \frac{\dot{V}_1}{V_2} = \ln \frac{\dot{I}_1}{I_2}
\]

Propagation constant is complex value and consists on slope (attenuation) coefficient \( \alpha \) and phase coefficient \( \beta \):

\[
g = \alpha + j\beta.
\]

In some textbooks propagation constant real part \( \alpha \) is called damping constant and imaginary part \( \beta \) – phase constant.

If four-terminal network is non-symmetric when \( \frac{\dot{V}_1}{V_2} \neq \frac{\dot{I}_1}{I_2} \). Propagation constant in such network could be found in following way:

\[
g = \frac{1}{2} \ln \frac{\dot{V}_1}{V_2} + \frac{1}{2} \ln \frac{\dot{I}_1}{I_2} = \frac{1}{2} \ln \frac{\dot{V}_1\dot{I}_1}{V_2I_2} = \ln \left( \sqrt{t_{11}t_{22}} + \sqrt{t_{12}t_{21}} \right)
\]

Production \( \dot{V}\dot{I} \) do not have physical mean in this equation. Earlier we had learned that its modulus is equal to total power:

\[
|\dot{V}\dot{I}| = S.
\]

Substituting this expression into previous equation yields

\[
\alpha + j\beta = \frac{1}{2} \ln \frac{\dot{V}_1\dot{I}_1}{V_2I_2} = \frac{1}{2} \ln \frac{V_1I_1}{V_2I_2} e^{i\phi} = \frac{1}{2} \ln \frac{S_1}{S_2} e^{i\phi} = \frac{1}{2} \ln \frac{S_1}{S_2} + j\frac{1}{2} \phi.
\]

Here \( \phi \) - difference in phases between voltage and current phasors. Damping constant

\[
\alpha = \frac{1}{2} \ln \frac{S_1}{S_2}
\]

could be measured in nepper units (Np). It is known that in the symmetric networks
\[ \alpha = \frac{1}{2} \ln \frac{V_1 I_1}{V_2 I_2} = \ln \frac{V_1}{V_2} = \ln \frac{I_1}{I_2}. \]

In practice more convenient is to use logarithms with decimal base. Damping constant can be measured in decibels as magnitude response:

\[ \alpha = 10 \cdot \log \frac{S_1}{S_2} \]

or

\[ \alpha = 20 \cdot \log \frac{V_1}{V_2} = 20 \cdot \ln \frac{I_1}{I_2}. \]

Primary parameters of the network may be expressed through characteristic impedance and propagation constant. Recalling previous equations

\[ e^g = \sqrt{t_{11} t_{22}} + \sqrt{t_{12} t_{21}}, \]

and doing simple transforms

\[ e^{-g} = \frac{1}{\sqrt{t_{11} t_{22}} + \sqrt{t_{12} t_{21}}} = \frac{\sqrt{t_{11} t_{22}} - \sqrt{t_{12} t_{21}}}{t_{11} t_{22} - t_{12} t_{21}} = \sqrt{t_{11} t_{22}} - \sqrt{t_{12} t_{21}}; \]

we are obtaining formulas necessary to express primary four-terminal networks parameters using their secondary parameters:

\[ \frac{e^g + e^{-g}}{2} = \text{chg} = \sqrt{t_{11} t_{22}}; \quad \frac{e^g - e^{-g}}{2} = \text{shg} = \sqrt{t_{12} t_{21}}; \quad Z_{01} Z_{02} = t_{12} \frac{Z_{01}}{t_{21}} = \frac{Z_{02}}{Z_{01}} t_{22}. \]

From the analysis of these equations we can obtain following expressions of the \( t \)-parameters:

\[ t_{11} = \sqrt{Z_{01} Z_{02}} \text{chg}; \quad t_{12} = \frac{Z_{01} Z_{02}}{t_{21}} \text{shg}; \quad t_{21} = \frac{1}{\sqrt{Z_{01} Z_{02}}} \text{shg}; \quad t_{22} = \frac{Z_{02}}{Z_{01}} \text{chg}. \]

Therefore, four-terminal network equations can be rewritten in following form:

\[ \begin{align*}
\dot{V}_1 &= \sqrt{Z_{01} Z_{02}} \left( \dot{V}_2 \text{chg} + Z_{02} \dot{I}_2 \text{shg} \right), \\
\dot{I}_1 &= \sqrt{Z_{01} Z_{02}} \left( \frac{\dot{V}_2}{Z_{02}} \text{shg} + \dot{I}_2 \text{chg} \right).
\end{align*} \]

If characteristic impedances \( Z_{01} \) and \( Z_{02} \) are the same (\( Z_{01} = Z_{02} = Z_0 \)), then
\[ \dot{V}_1 = \dot{V}_2 \text{chg} + Z_0 \dot{I}_2 \text{shg}, \]
\[ \dot{I}_1 = \frac{\dot{V}_2}{Z_0} \text{shg} + \dot{I}_2 \text{chg}. \]

Later, in the section ____, we’ll show the same nature of equations for the networks with distributed parameters.